GENERAL RELATIVITY

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A. The theory

I. Six heuristic requirements : the six principles of GR

1. The relativity principle
2. The equivalence principle
3. The correspondence principle
4. The conservation principle
5. The minimal coupling principle
6. Mach's principle

helped Einstein in the search for the correct theory.
Equivalence principle extraordinary

1. Relativity principle : all observers are equivalent

SR ~~inertial~~ observers. Now all.
Any coordinate system. Theory invariant under coord. transf. => General covariance principle : eqs. of physics have a tensorial form

2. Equivalence principle : the effects of acceleration are indistinguishable from effects of gravity, locally!

3. Correspondence principle : GR -> SR and GR -> Newtonian gravitation.
One in absence of gravitation, the other for weak grav. fields and low velocities.

4. Conservation principle : $\nabla_b T^{ab} = 0$

compatibility of the field equations with the energy-momentum tensor conservation.

5. Minimal coupling principle : no unnecessary terms from SR $\rightarrow$ GR

$\partial_b T^{ab} = 0 \longrightarrow \nabla_b T^{ab} = 0$ (could also be $\nabla_b T^{ab} + g^{be} R^a{}_{bcd} \nabla_e T^{cd} = 0$)

6. Mach's principle : matter distribution determines geometry. GR with precise boundary conditions may obey it !

Equivalence principle . seems trivial

How such a triviality led to GR ?

Books do not distinguish inertial m and gravitational m

One is $m = m_i$ other charge $q = q_g = m_g$

Newton's second law $F = ma$

F : push, electrical, gravitational. Latter

$F = q_g g$ (or $F = q_g \dfrac{G q_{g\,earth}}{r^2}$) . so

$q_g g = m a \implies a = \dfrac{q_g}{m} g$

Compare a_s of 2 bodies : equal $\Rightarrow a_1 = a_2 \Rightarrow \dfrac{q_g^1}{m_1} = \dfrac{q_g^2}{m_2}$

valid for any other body so $\dfrac{q_g^1}{m_1} = \text{const} = 1$

$\implies \quad q_g = m$ In Newt. derived from exp.

A coincidence.

Implies $a = g$

(for other forces does not hold, e.g., $q = q_e$)

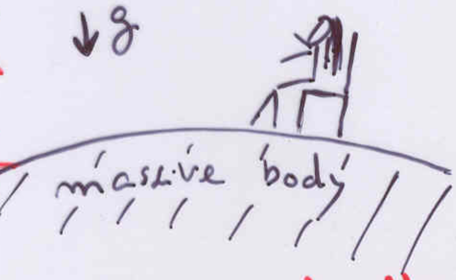
$a = g \Rightarrow$ Enunciate 0: The motion of a test ptle in a g-field is independent of its mass and composition

Known to John Philoponus of Alexandria (~520)
Simon Stevin of Bruges (~1590)
Galileo of Pisa (~1600)

Two rocks left from any tower: one sees the other hovering stationary. Earth's gravity seems to have vanished. Essence of the principle. Einstein 1907 (then 1912).

Paradigmatic Examples

• 2 observers in vicinity of massive body. One in free fall the other resisting grav. Both can claim to be at rest if they agree to disagree about presence of g field.



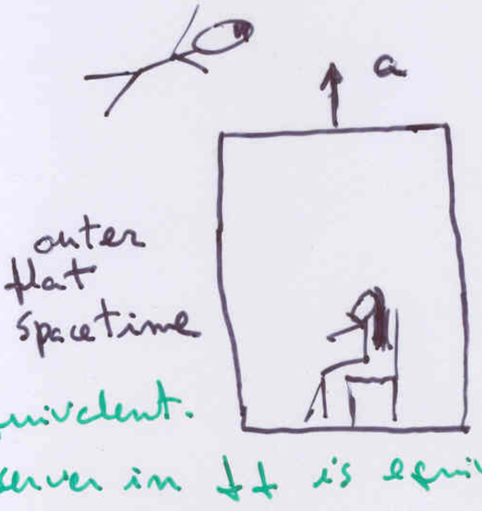
Freefall: no g field, he is at rest, other acc.

Resisting: there is g field, she is at rest, other acc.

• 2 observers in flat region.
One hovering freely, the other accelerated in the spaceship.

Freefall: same

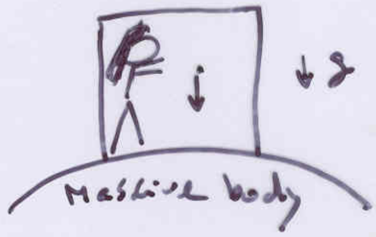
Spaceship: same



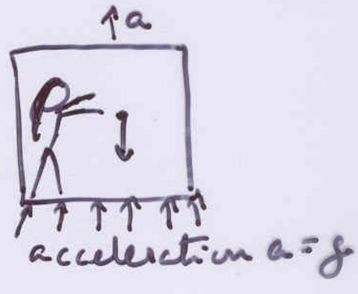
The two observers are not equivalent.
What is equivalent? Observer in ff is equivalent to observer hovering. Observer resisting is equivalent to observer accelerated in ship.
The presence or absence of g-field is relative, not the motion.

Further experiments on a closed spaceship (modern elevator)

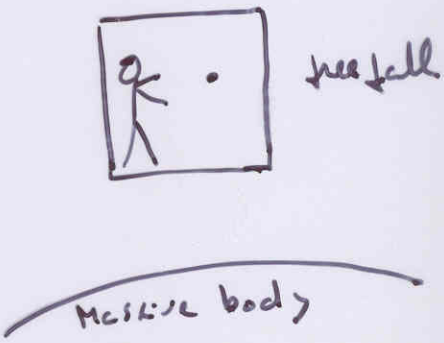
Experiment 1: Ship placed at rest, its horizontal on surface of massive body. Ball released, falls with acceleration g to the floor.



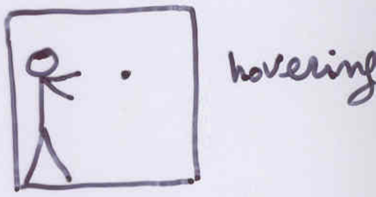
Experiment 2: ship placed on a rocket far from bodies, accel. upwards $a = g$. Ball released, falls with acceleration g to the floor.



Experiment 3: ship is placed in free fall near massive body. The ball falls with ship and observer. Remains at rest relative to him.



Experiment 4: ship far from massive bodies, not acc, hovering. The ball remains at rest.



Enunciate 1: A frame at rest in a g-field is locally identical to a frame linearly accelerated relative to an inertial frame in SR.

Enunciate 2: There are no local experim. which can distinguish free fall (non rotating) in a g-field from uniform motion in space (with a zero g-field).

Appropriate mathematical formalism, generalise from SR. Coord. system adapted to inertial frame, eq. of motion of free ptles

$$\frac{d^2 x^a}{d\tau^2} = 0 \qquad \tau \text{ proper time}$$

Arbitrary coordinate system leads to

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

Γ^a_{bc} are the inertial terms $\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc})$

and g_{ab} flat but not Minkowski.

Through P. of equivalence both inertial forces and g-forces are in Γ^a_{bc}. Then ST may be no longer Minkowskian of SR!

Simplest, Riemannian $\Rightarrow$ GR

(g_{ab} g-potential and inertial structure, Γ^a_{bc} g-force, R^a_{bcd} tidal force)

But equivalence principle does not eliminate gravity? Can't we always eliminate gravity choosing a free fall frame?

Subtleties that lead to the tidal force.

II Gravitation and Curvature

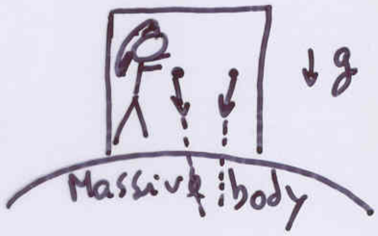
Local analysis, ignored variation of g-field from place to place.

Down here not the same as in Lisbon. Frame freely falling here does not do the job there.

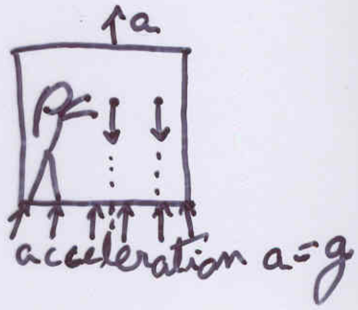
Elimination of g-field not straightforward!

Huge space ship

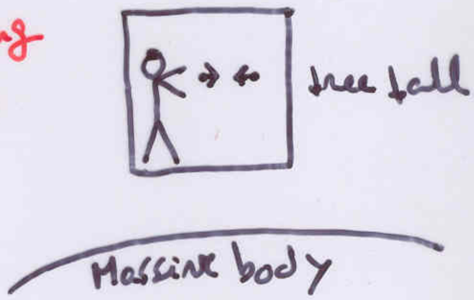
Experiment 1: with the ship on the surface of massive body the 2 balls fall towards center, on converging lines.



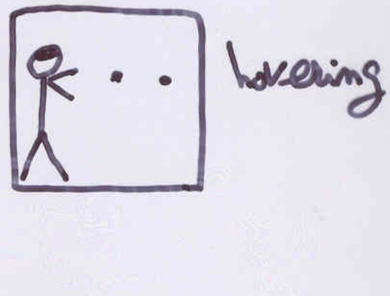
Experiment 2: Releasing the 2 balls at nearby points the two balls accelerate donward on parallel lines



Experiment 3: With the elevator falling freely, the 2 balls move towards one another (not because of mutual attraction).



Experiment 4: In outer space, with rocket motors off, the 2 balls remain at rest inside the ship after their release.

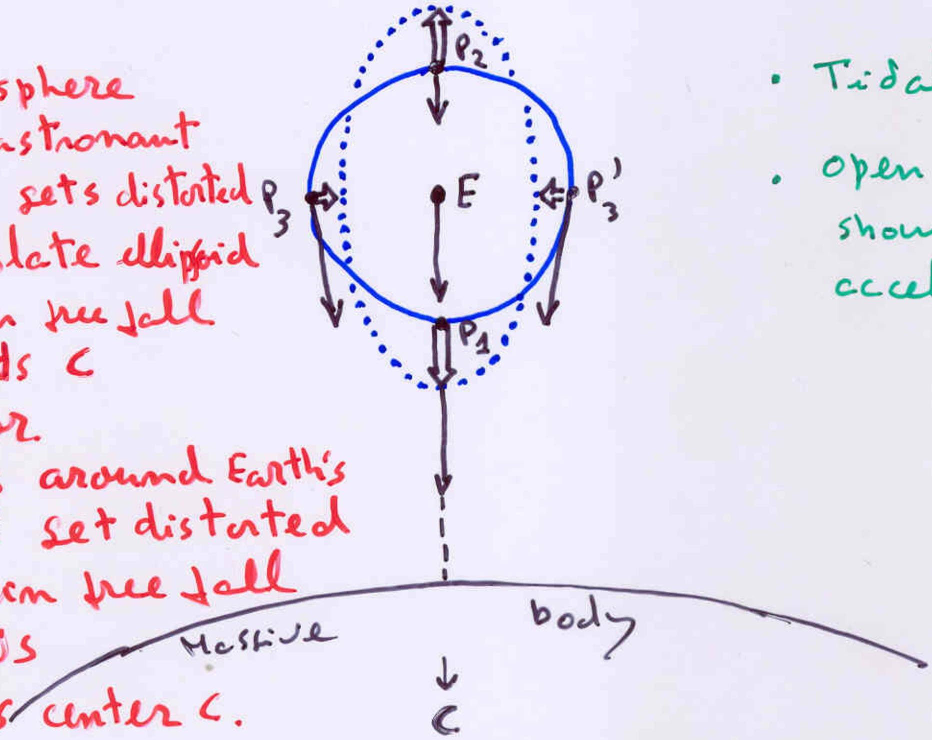


Distortion of ptles is tidal effect of gravity (Earth's tides)

- small sphere around astronaut E(dward) gets distorted into prolate ellipsoid when in free fall towards C

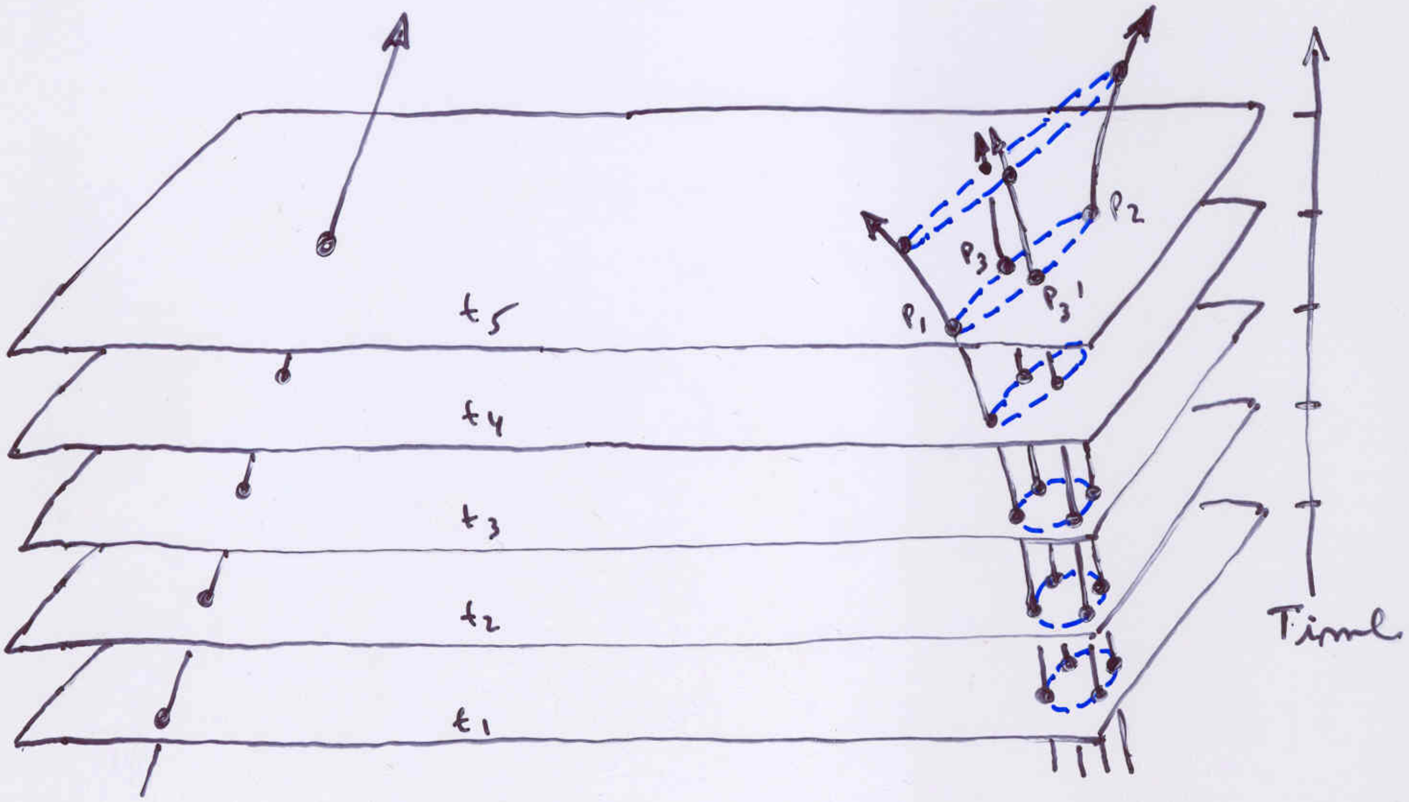
or

- Oceans around Earth's center E get distorted when in free fall towards moon's center C.



- Tidal effect
- open arrows show relative acceleration

Spacetime picture of tidal effect: geodesic deviation



Bending effect in ST: lines curve due to gravitation

If geodesics curve $\Rightarrow$ curved space

Analogous for ST

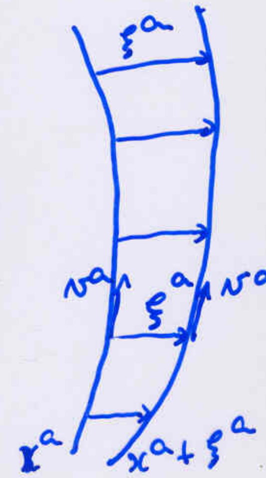
In SR $\quad ds^2 = \eta_{ab}\, dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2$

geodesics are straight lines, inertial ptles

Now, with equivalence principle, new concept of inertial $\Rightarrow$ free fall under gravity.

Tidal effects $\Rightarrow$ geodesic deviation $\Rightarrow$ curved ST

Formalizing:



Newtonian - 2 ptles $m=1$, coordinates

Move in potential ϕ $\qquad x^a(t)$ and $x^a(t)+\xi^a(t)$

Eq. of motion

$$\begin{cases} \ddot{x}^a = -\partial^a\phi \\ \ddot{x}^a + \ddot{\xi}^a = -(\partial^a\phi)_{x+\xi} = -\partial^a\phi - \xi^b\partial_b\partial^a\phi \end{cases}$$

sub $\Rightarrow \quad \ddot{\xi}^a + K^a{}_b\,\xi^b = 0 \qquad K^{ab} = \partial^a\partial^b\phi \quad \left(K^a{}_a = 0 \text{ is the Laplace eq.}\right.$

curved space(time) - curvature tensor (Riemann)

$R^a{}_{bcd}$ controls the getting close or apart

$$\ddot{\xi}^a + R^a{}_{bcd}\, v^b \xi^c v^d = 0$$

or $\sim \qquad \ddot{\xi}^a + \bar{R}^a{}_b\,\xi^b = 0 \qquad (\bar{R}^a{}_b = R^a{}_{bcd} v^c v^d)$

(so $\bar{R}^a{}_b$ is analogous to $K^a{}_b$ and $\bar{R}^a{}_b = 0$ should be Einstein's equation, analogous to Laplace).

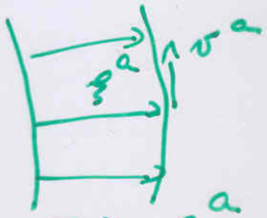
III Einstein's Equation

In ST Newtonian theory in vacuum there is no volume change of initial congruence. when matter inside original sphere volume reduction (contraction) $\propto$ mass (density)

Riemann tensor decomposed

$$R_{abcd} = \underbrace{W_{abcd}}_{\text{weyl}} + \underbrace{g_{ab}}_{\text{metric}} \underset{\text{Ricci}}{R_{cd}} + g_{ab} g_{cd} R$$

traceless conformal



For a congruence ξ^a variation of the volume expansion (or contraction) $\theta = \nabla_a \xi^a$

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{\sigma_{ab}}_{\text{shear}}\sigma^{ab} + \underbrace{\omega_{ab}}_{\text{twist}}\omega^{ab} - R_{ab}v^a v^b$$

(Raychaudhuri's equation)

see that ~~to~~ 1st order R_{ab} gives volume reduction

$\Rightarrow$ R_{ab} related to ρ (density)

$\Rightarrow$ T_{ab} energy-momentum

Try $R_{ab} = 4\pi G T_{ab}$ as Einstein did!

But $T_a{}^b{}_{;b} = 0$ and $R_a{}^b{}_{;b} \neq 0$. Change to

$R_{ab} - \frac{1}{2} R g_{ab} \equiv G_{ab}$ with $G_a{}^b{}_{;b} = 0$.

$$\Rightarrow \quad \boxed{G_{ab} = 8\pi G T_{ab}}$$

(Since then $R_{ab} = 8\pi G (T_{ab} + \frac{1}{2} g_{ab} T)$ volume change $\propto (\rho + 3P)$ rather than ρ as in Newtonian gravitation)

Tests in the solar system

- Gravitational redshift
- Precession of Mercury orbit around the sun
- Light bending (now lensing, in cosmology)
- Radar echo from planets
- Precession of gyroscopes in a rotating g-field, like the Earth.
- Global Positioning System (GPS), technological application.

GPS - determine your position on Earth in three dimensions (longitude, latitude, altitude).
24 satellites in circular orbits around Earth

- Each sends a signal codes where it is and time

Receiver clock times the reception, subtracts emission time, determines elapsed time and how far → gives distance from satellite to you. Needed 3 satellites. Draw a sphere on the emission point of each satellite → point of intersection is where you are.
(A 4th clock checks accuracy of both clocks)

- Signals exchanged by atomic clocks at different altitudes are subject to GR effects. Neglecting these would make the GPS useless.

• why? Use Schwarzschild metric $(G=c=1)$

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2M}{r}} - r^2 d\phi^2$$

τ proper time of clock

t coord. time (far away time)

$dr = 0$ for clock on Earth and for clock on satellite

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - v^2$$

$v = r\frac{d\phi}{dt}$ tangential velocity with respect to far away time.

$$\frac{d\tau_{satellite}}{d\tau_{Earth}} = \left(\frac{1 - \frac{2M}{r_{satellite}} - v^2_{satellite}}{1 - \frac{2M}{r_{Earth}} - v^2_{Earth}}\right)^{1/2}$$

$$\simeq 1 - \frac{M}{r_{satellite}} - \frac{v^2_{satellite}}{2} + \frac{M}{r_{Earth}} + \frac{v^2_{Earth}}{2}$$

$r_{satellite} = 26.6 \times 10^3$ km $\quad r_{Eart} = 6.4 \times 10^3$ km

$M = 5.9 \times 10^{24}$ kg $= 0.444$ cm

So $\frac{M}{r_{Earth}} \simeq 10^{-9}$ Important?

In 1 nanos light travels 30 cm

so difference of 100 nanos $\Rightarrow$ 30 m $\Rightarrow$ difficulty
in clocks

1 day $\simeq 10^5$ s $\Rightarrow \Delta\tau_{satellite} - \Delta\tau_{Earth} \simeq 10^{-9}\,\Delta\tau_{Earth}$
$\simeq 10^5$ nanos

$\Rightarrow$ Have to take into account the GR and SR effects

B. Applications

I. Black Holes

1. Importance

- ultimate fate of massive stars ($50 M_\odot$)
  (Astrophysical)
- ultimate fate of centers of galaxies ($10^8 M_\odot$)
  (Astrophysical)
- Gift of mother nature, primordial Universe
  (cosmological)
- collision of ptles in accelerators (large extra dimensions scenario)
  (particle physics)
- pair creation (particle physics, fundamental physics)

Classical aspects : horizon, singularity, no hair $\rightarrow$ astrophysics

Quantum aspects : entropy, thermodynamics, ptle creation $\rightarrow$ quantum gravity, fundamental

Huge subject : see "Black Hole Physics" by Frolov and Novikov, 800 pages and is a summary (published 2000)

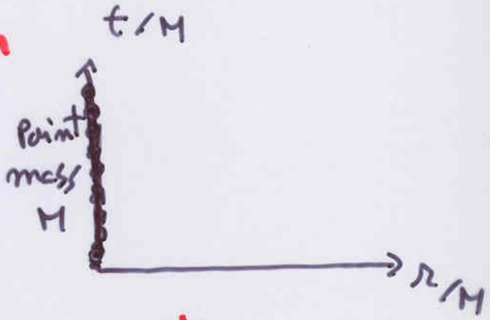
select one ! : Geometry and Dynamics

2. Geometry and dynamics of the Schwarzschild solution

a. Schw. solution in Schw coord.

- Newtonian Gravitation: $\nabla^2\phi = 4\pi G\rho \xRightarrow{\text{vacuum}}$ $\nabla^2\phi = 0$. Sph. Sym. solution

$$\phi = -\frac{GM}{r}$$ point mass M at origin ($r=0$)



In the ST diagram every point represents a sphere.

- GR: $G_{ab} = 8\pi G T_{ab} \xRightarrow{\text{vacuum}} G_{ab} = 0$ $\left(\frac{\partial^2 g_{ab}}{\partial x^c \partial x^d} + \dots = 0\right)$
  - sph. sym. (G=1) $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{d\Omega^2}$

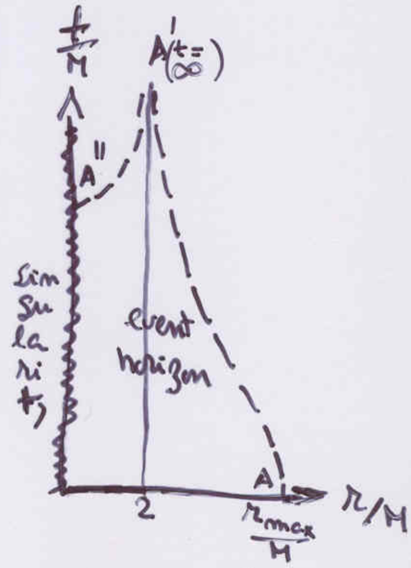
    what does it represent? Point mass?
  - Problem at $r = 2M \to 0$ and ∞! but $|R_{abcd}| \sim \frac{M}{r^3} \sim \frac{1}{M^2}$

    so geometry is fine.

    coord. system has a (physical!) pathology!
  - $r < 2M$ t is spatial and r is time!
  - Radial geodesics

$$r = \frac{1}{2} r_{max}(1+\cos\eta) \Rightarrow \pm\left(\frac{r_{max}}{r} - 1\right)^{1/2} = \tan\frac{\eta}{2}$$
$$-\pi \le \eta \le \pi$$

two times
$$\left\{\begin{aligned} \tau &= \left(\frac{r_{max}^3}{8M}\right)^{1/2}(\eta + \sin\eta) \quad \text{(proper)} \\ t &= 2M \ln\left|\frac{\left(\frac{r_{max}}{2M}-1\right)^{1/2} + \tan\eta/2}{\left(\frac{r_{max}}{2M}-1\right)^{1/2} - \tan\eta/2}\right| + f(\eta) \quad \text{(far away)} \end{aligned}\right.$$



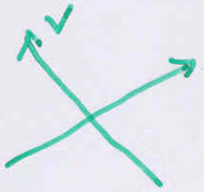
- Conclude
  - For faraway t geodesic takes ∞ time to arrive at $r = 2M$
  - $r = 2M$ is special, null geodesic
  - For proper time τ, it takes $\tau_{r=0} = \left(\frac{r_{max}^3}{8M}\right)^{1/2}\pi$
  - $r = 0$ singularity of ST, nasty
  - No resemblance to point mass at all!

b. Schw. solution in Kruskal coordinates

. start now with null coordinates U, V

$$ds^2 = -2B(U,V)\,dU\,dV + r^2(U,V)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

null because for radial geod $\begin{cases} U = \text{const} \\ \text{or } V = \text{const} \end{cases} \Rightarrow ds = 0$



solve $G_{ab} = 0$ $\left(\frac{\partial^2 g_{ab}}{\partial x^c \partial x^d} + \dots = 0 \right)$

$-\infty < U < \infty$
$-\infty < V < \infty$

find

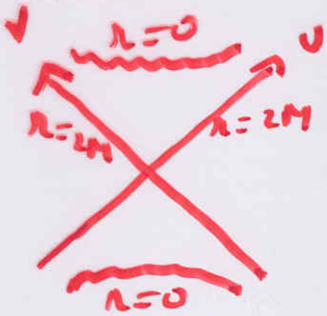
$$ds^2 = -\frac{16M^3}{r}\exp\left(-\frac{r}{2M}\right) dU\,dV + r^2(U,V)\, d\Omega^2$$

with $r(U,V)$: $-UV = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$

• Note: . no problem at $r = 2M$ in ds

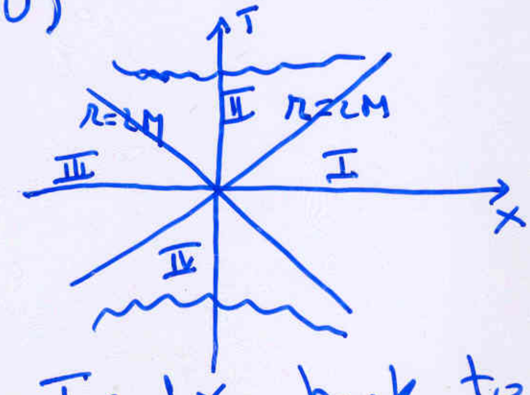
. $r = 2M$ $UV = 0$
$\Rightarrow U = 0$ or $V = 0$

. $r = 0$ $UV = 1$
$\Rightarrow$ singularity is two hyperbolae



• Change back to time T and space X (rather than null)

$T = \frac{1}{2}(V + U)$
$X = \frac{1}{2}(V - U)$ $\longrightarrow$ $T^2 - X^2 = UV$



can relate T and X back to t, r (Schw)

- relation $(t,r) \to (T,x)$

$$\mathrm{I}\quad \begin{cases} T = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M} \sinh(t/4M) \\ x = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M} \cosh(t/4M) \end{cases}$$

$$\mathrm{II}\quad \begin{cases} T = \left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh(t/4M) \\ x = \left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh(t/4M) \end{cases}$$

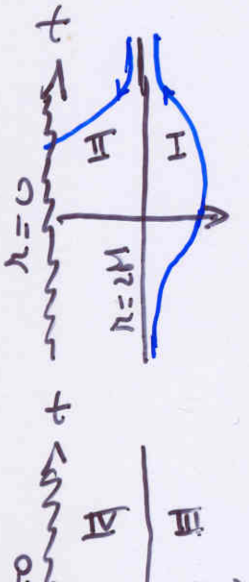
In $\underline{\mathrm{III}}$ and $\underline{\mathrm{IV}}$ regions with $-$signs

inverse $(T,x) \to (t,r)$

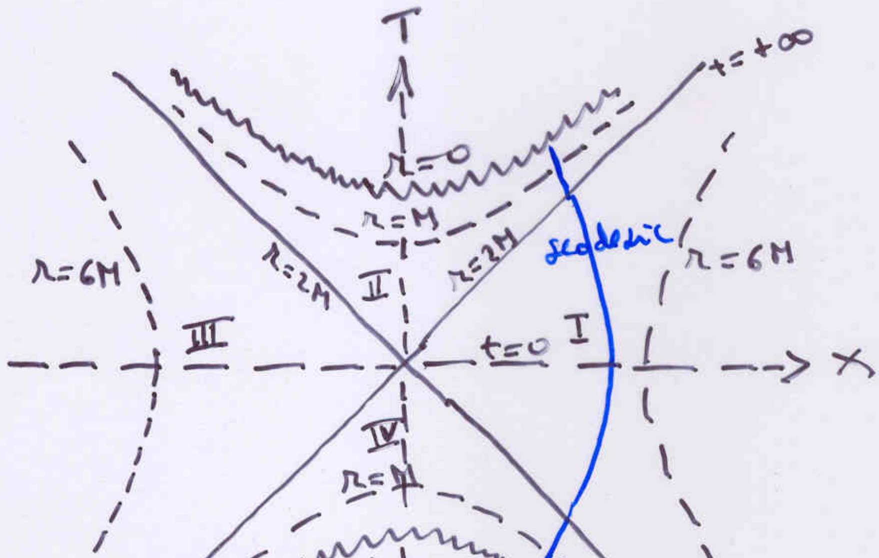
$$t = \begin{cases} 4M \tanh^{-1}\left(\frac{T}{x}\right) & \text{in I and } \underline{\mathrm{III}} \\ 4M \tanh^{-1}\left(\frac{x}{T}\right) & \text{in II and } \underline{\mathrm{IV}} \end{cases}$$

$$\left(\frac{r}{2M}-1\right)^{1/2} e^{r/2M} = x^2 - T^2 \quad \text{in I, } \underline{\mathrm{II}}, \underline{\mathrm{III}} \text{ and } \underline{\mathrm{IV}}$$

- Diagrams



Schwarzschild patches



Kruskal-Szekeres

- Point mass in GR is totally different from point mass in Newt. gravitation! Richness of causal structure never seen.
  What does it represent physically?

6. Interpretation

- Schw. solution (a vacuum solution) is a Wheeler wormhole: white hole, black hole, and two asymptotic regions connected through a (non-traversable) wormhole.

- Up to now examined (t, r) and (T, X) Kruskal planes. Also have to underst.
- spatial sections (r, θ, ϕ) for T or t fixed. Say $T=0$ or $t=0$

since sph. symm. $(\theta, \phi) \to \phi$ 

$$t=0\,,\ \theta=\frac{\pi}{2} \Rightarrow ds^2 = \frac{dr^2}{1-2M/r} + r^2 d\phi^2$$

flat as $r \to \infty$, $r \to 2M$?

Embed in 3D flat space $ds^2 = dz^2 + dr^2 + r^2 d\phi^2$

$z = z(r)$

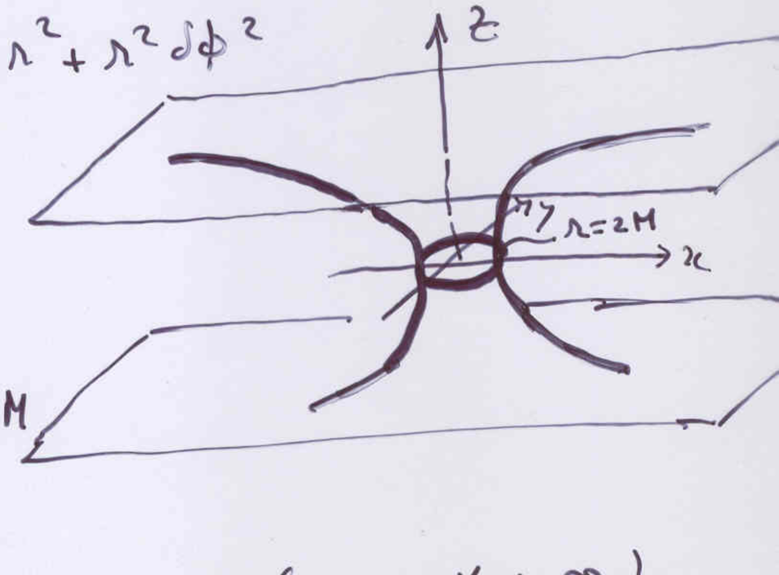
- $ds^2 = \left[\left(\frac{dz}{dr}\right)^2 + 1\right] dr^2 + r^2 d\phi^2$

$$\Downarrow$$

$$\left(\frac{dz(r)}{dr}\right)^2 + 1 = \frac{1}{1-\frac{2M}{r}}$$

$$\Downarrow$$

$$z(r) = \pm\left(\frac{r}{2M} - 1\right)^{1/2} 4M$$

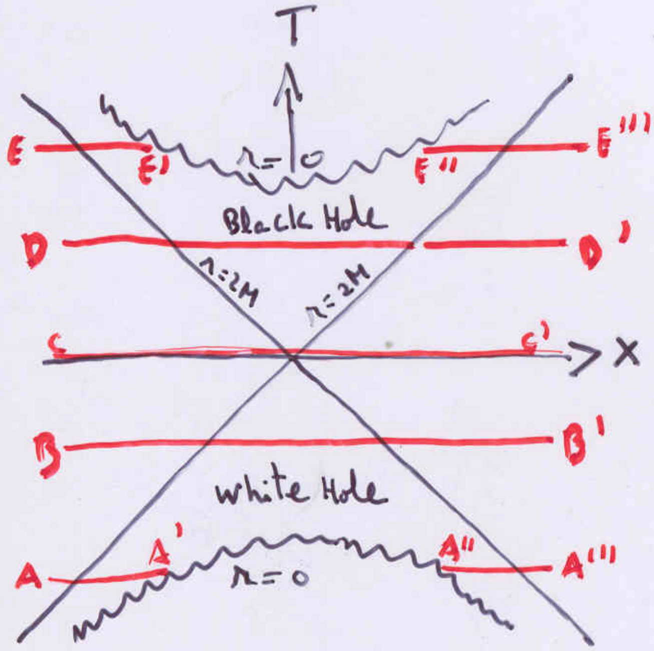


or $r = 2M + \frac{z^2}{8M}$

$\Rightarrow$ Geometry at $t=0$ $(T=0)$ $(-\infty < X < \infty)$ is a bridge or wormhole connecting two distinct asymptotically flat worlds.
(Einstein-Rosen bridge, Schw throat, Schw. wormhole)

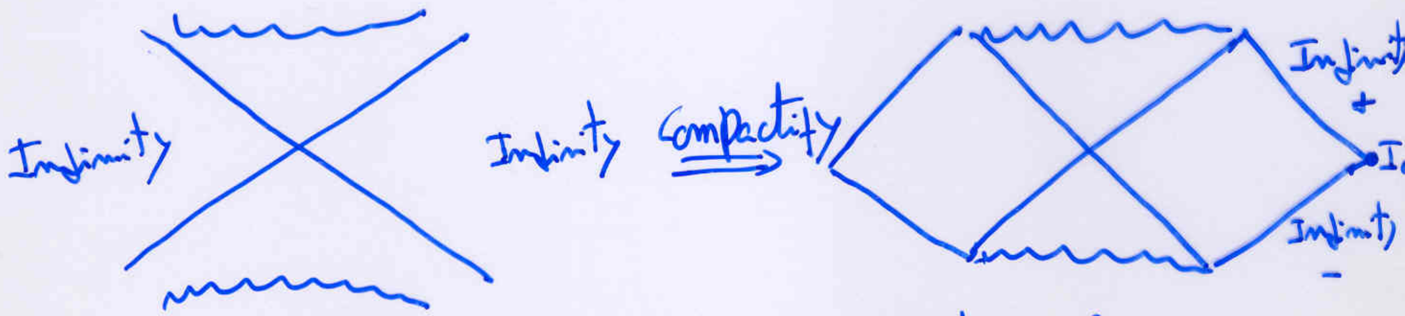
what happens as the $t=0$ ($T=0$) hypersurface is push forward in time T. Geometry changes (it depends on time).

Wheeler Wormhole



- Evolution: first two disconnected universes each with a singularity, then the singularities join forming a wormhole, maximum throat is $r=2M$, then contracts, pinches off, leaving two universes
- Formation, expansion and collapse is so rapid, no light ray passes over the bridge.
- From Kruskal and 45° nature of light rays, ptle in region IV (white hole) must have been created at singularity; any particle that falls into region II (black hole) is doomed
- Wheeler wormhole (dynamic) is the point mass solution of GR! (totally different from Newt. gravitation) (1916 – 1960)

d. Carter-Penrose Diagram

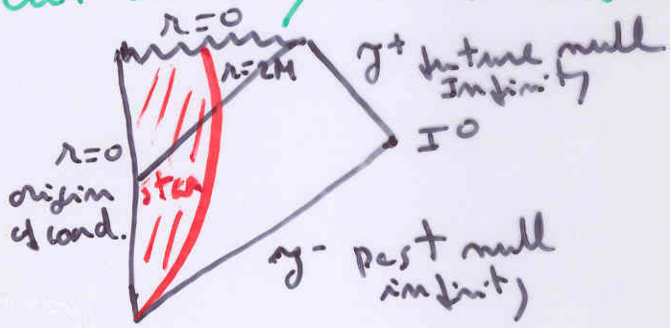


Kruskal (schw. solution)

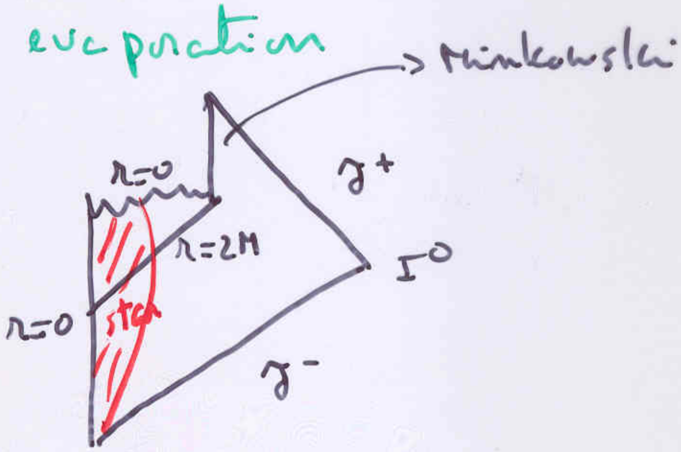
Carter-Penrose (schw. solution)

e. Black Holes from gravitational collapse and Hawking radiation

- collapse: with a star as interior the schw solution only works outside



- Hawking evaporation



II Gravitational Radiation

1. Introduction

gravitational waves : small perturbations on the background spacetime (flat)

- Wave equation $\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$ $\quad f = A\cos(\omega t - kx)$
  $\omega^2 = k^2 c^2$

- transverse plane waves : travelling with c (≃ EM waves)

- Spacetime structure oscillates (proper time that light takes between 2 points oscillates)

- There is no effect in a single point - only variation in the distance between 2 points (contrast with EM, charges oscillate in the point where the wave is passing)

- Dominant waves are quadropolar (EM dipolar)

- $\frac{c^4}{8\pi G}$ is the force per unit area required to give ST unit curvature (recall $G_{ab} = \frac{8\pi G}{c^4} T_{ab}$)
  $\simeq 10^{43}\, N/m^2$ for a curvature of $1\,m^{-2}$ $\Rightarrow$ ST is a stiff medium

- Events : star collapse, coalescence
  Produce deformations in amplitude $< 10^{-18}$

- Detectors : interferometers, resonant bars

- Indirect observation : binary pulsar : 1913+16 and companion. Orbital period decreases at a rate predicted by GR.

2. Properties of gravitational radiation

vacuum $G_{ab} = 0 \implies R_{ab} = 0$

metric $g_{ab} = \underbrace{\eta_{ab}}_{\text{Minkowski}} + \underbrace{h_{ab}}_{\ll 1}$

$$R^{(1)}_{bd} = \frac{1}{2}\left(h^{a}{}_{d,ba} - h_{db}{}^{,a}{}_{,a} + h_{ab}{}^{,a}{}_{,d} - h^{a}{}_{a,bd}\right) = 0$$

choose : $\underbrace{h^{a}{}_{a} = 0}_{\text{traceless}}$ $\underbrace{h_{ab,}{}^{a} = 0}_{\text{zero divergence}}$ and $h_{a0} = 0$ $\left\{\begin{array}{l}10 \text{ components} \\ 8 \text{ constraints}\end{array}\right. \Rightarrow$ 2 indep. comp.

Equation: $h_{bd,}{}^{a}{}_{a} = 0$ or $\dfrac{\partial^2 h_{bd}}{\partial x^a \partial x^a} = 0$ or

$$\underbrace{\frac{\partial^2 h_{bd}}{\partial (ct)^2} - \frac{\partial^2 h_{bd}}{\partial x^2} - \frac{\partial^2 h_{bd}}{\partial y^2} - \frac{\partial^2 h_{bd}}{\partial z^2} = 0}_{\text{wave equation}}$$

solution: $h_{bd} = A_{bd}\, e^{i k_a x^a}$

wave in z-direction $k_a = (k_0, 0, 0, k)$

so $k^a k_a = 0 = k^{0\,2} - k^2 = 0$

$\Rightarrow h_{bd} = A_{bd} \cos(\underbrace{kc}_{\omega} t - kz)$

other conditions : $h_{ab,}{}^{a} = \dfrac{\partial h_{ab}}{\partial x^a} = A_{ab}\, i k^a = 0$

$\Rightarrow k^a A_{ab} = 0 \implies k^0 A_{0b} + k^3 A_{3b} = 0 \implies A_{0b} + A_{3b} = 0$

but $h_{b0} = 0 \implies A_{3b} = 0 \Rightarrow$ only transverse x, y components

$h^{a}{}_{a} = 0 \implies A_{11} + A_{22} = 0$

Also symmetry $\implies A_{12} = A_{21}$

so $(A_{ab}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ amplitudes A_{ab} are transverse traceless

It is the gauge T T

There are 2 independent states

$$(e_+)_{bd} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(e_\times)_{bd} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Any $A_{bd} = A_+ (e_+)_{bd} + A_\times (e_\times)_{bd}$

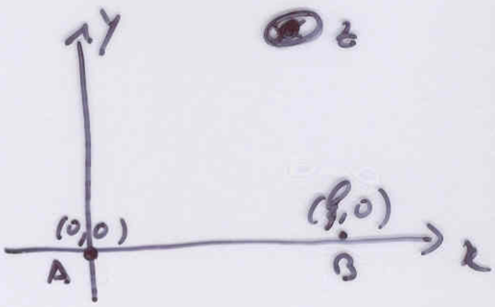
$$h_{bd} = A_+ (e_+)_{bd} \cos(\omega t - kz) + A_\times (e_\times)_{bd} \cos(\omega t - kz)$$

$$g_{bd} = \eta_{bd} + h_{bd}$$

3. Effects of gravitational radiation

2 ptles at $z=0$ when a wave passes in the z direction

ptles along x



$$\bar{\xi} = \sqrt{g_{11}(t,0)}\, \xi = \left[1 + \frac{1}{2} h_{11}(t,0)\right] \xi$$

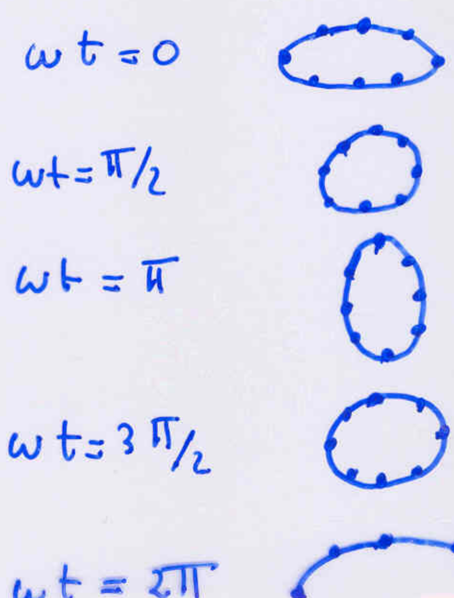
$$\Rightarrow \Delta \xi = \bar{\xi} - \xi = \frac{h_{11}}{2} \xi = \frac{h_+}{2} \xi \Rightarrow \varepsilon = \frac{\Delta \xi}{\xi} = \frac{h_+}{2}$$

2 ptles along y $\quad \varepsilon = -\frac{h_+}{2}$

$$\begin{cases} x = x_0 \left(1 + \frac{h_+}{2} \cos\omega t\right) \\ y = y_0 \left(1 - \frac{h_+}{2} \cos\omega t\right) \end{cases}$$

quadrupole motion

e_+ polarization



analogous for $e_\times$ polarization

4. Energy and Flux

GWs carry energy $\Rightarrow t_{ab}$ (stress-energy of the waves)

$$\left\{\begin{array}{ll} & \Rightarrow \quad G^{(1)}_{ab} = \dfrac{8\pi G}{c^4} t_{ab} \\ \text{vacuum} & G^{(1)}_{ab} + G^{(2)}_{ab} = 0 \end{array}\right. \Rightarrow t_{ab} = -\frac{c^4}{8\pi G} G^{(2)}_{ab}$$

Find $t_{00} = \dfrac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$ (energy density)

Flux $F = t_{00} c = \dfrac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle = \dfrac{c^3}{32\pi G} \langle \dot{h}^{ij} \dot{h}_{ij} \rangle$

$\langle \quad \rangle$ average over cycles and wavelengths

Quasi-Newtonian sources: $\left\{\begin{array}{l} \cdot \text{ small curvature} \\ \cdot \ v \ll c \end{array}\right.$

From $\dfrac{\partial^2 h_{ab}}{\partial x^m \partial x_m} = \dfrac{16\pi G}{c^4} T_{ab}$

Find $h_{ij}(t) = \dfrac{2G}{c^4} \dfrac{1}{r} \ddot{I}^{TT}_{ij}(t - r/c)$

I^{TT}_{ij} quadrupole moment of source $\left(\begin{array}{l}\text{transverse} \\ \text{traceless}\end{array}\right)$

$t - \frac{r}{c}$ retarded: amplitude at r at t determined by source behavior at earlier time $t - \frac{r}{c}$

so $F = \dfrac{G}{8\pi r^2 c^5} \langle \dddot{I}^{TT}_{ij} \dddot{I}^{TT}_{ij} \rangle$

$L = \dfrac{G}{5c^5} \langle \dddot{\cancel{I}}_{ij} \dddot{\cancel{I}}_{ij} \rangle \qquad \cancel{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$

Example: axisymmetric collapse of star

orders of magnitude

quadropole moment of inertia $I \simeq MR^2$

star oscillates with ω

$$\Rightarrow \quad \dddot{I}_{ij} \simeq MR^2\omega^3 \simeq M v^3/R$$

$$\Rightarrow \quad L = \frac{G}{5c^5} \langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle \simeq \frac{G}{c^5} \frac{M^2 v^6}{R^2}$$

$$\simeq \frac{c^5}{G} \left(\frac{GM/c^2}{R}\right)^2 \left(\frac{v}{c}\right)^6$$

$$\simeq L_0 \left(\frac{R_s}{R}\right)^2 \left(\frac{v}{c}\right)^6 \qquad R_s \simeq \frac{2GM}{c^2} \text{ schw. radius}$$

$$L_0 \equiv \frac{c^5}{G} = 3.63 \times 10^{59} \text{ erg/s} \qquad \left\{ \begin{array}{l} v \sim c \\ R \sim R_s \end{array} \right. \Rightarrow L = L_0 !$$

Generators: . rotating metal of 1000 tons with $v \sim 10^3$ m/s $\Rightarrow L \sim 10^{-23}$ erg/s

. Type II supernova $\frac{R_s}{R} \sim 10^{-3} \quad \frac{v}{c} \sim 1$

$$\Rightarrow L \sim 10^{53} \text{ erg/s}$$

Detection on Earth: suppose Δm of star is converted into GW

$$\Delta m\, c^2 = 4\pi r^2 \int F\, dt = \frac{c^3}{8G} r^2 \omega_0^2 h_0^2 \Delta\tau$$

$$\Rightarrow \quad h_0 = 10^{-18} \left(\frac{1\,\text{kHz}}{\nu}\right)\left(\frac{10\,\text{kpc}}{r}\right)\left(\frac{\Delta m}{10^{-3} M_\odot}\right)\left(\frac{1\,\text{ms}}{\Delta\tau}\right)^{1/2}$$

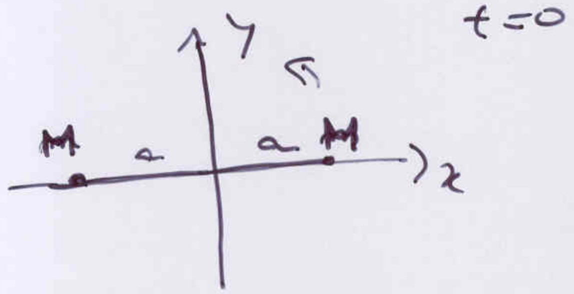
$$h_0\left(\begin{array}{c}\text{galactic}\\ \text{SN}\end{array}\right) \sim 10^{-18} \qquad h_0\left(\begin{array}{c}\text{virgo}\\ \text{SN}\end{array}\right) \sim 10^{-22}$$

$$\Downarrow$$

detectors

5. Binary pulsar 1913+16 and companion

$$\tau = 7.75\,\mathrm{h} = 27\,906.98163(2)\,\mathrm{s}$$
$$a \sin i = 2.34185(12) \text{ light sec.}$$
$$e = 0.617127(3)$$
$$M = 1.42(3)\,M_\odot$$
$$M' = 1.40(3)\,M_\odot \qquad \text{(Taylor)}$$
$$\sin i = 0.76(14)$$
$$\frac{d\tau}{dt} = -2.40(9) \times 10^{-12}\,\mathrm{s\,s^{-1}}$$



Consider two point masses separated by 2a in circular orbits

$$I_{xx} = 2Mx^2 = 2Ma^2 \cos^2\omega t = Ma^2(1+\cos 2\omega t)$$
$$I_{yy} = 2My^2 = 2Ma^2 \sin^2\omega t = Ma^2(1-\cos 2\omega t)$$
$$I_{xy} = I_{yx} = 2Ma^2 \cos\omega t \sin\omega t = Ma^2 \sin 2\omega t$$
$$\cancel{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$$
$$\cancel{I}_{xx} = I_{xx} - Ma^2 = Ma^2 \cos 2\omega t$$
$$\cancel{I}_{yy} = I_{yy} - Ma^2 = -Ma^2 \cos 2\omega t$$
$$\cancel{I}_{xy} = I_{xy} = Ma^2 \sin 2\omega t = \cancel{I}_{yx}$$

$$L = \left(\frac{G}{5c^5}\right)\langle \dddot{\cancel{I}}_{ij}\, \dddot{\cancel{I}}_{ij} \rangle = \left(\frac{G}{5c^5}\right)(2\omega)^6 (Ma^2)^2 \times \langle 2\sin^2 2\omega t + 2\cos^2 2\omega t\rangle$$
$$= \frac{G}{5c^5}\left(128\,\omega^6 M^2 a^4\right)$$

- rate of energy loss :

- $E = Mv^2 - \frac{GM^2}{2a}$ (total energy of the pair)

$$M\frac{v^2}{a} = \frac{GM^2}{4a^2} \Rightarrow v^2 = \frac{GM}{4a}$$ (eq. of motion of each star)

$$\Rightarrow \omega^2 = \left(\frac{v}{a}\right)^2 = \frac{GM}{4a^3}$$

$$\Rightarrow E = -\frac{GM^2}{4a} = -\frac{GM^2}{4}\left(\frac{4\omega^2}{GM}\right)^{1/3}$$

$$\Rightarrow \frac{dE}{E} = \frac{2}{3}\frac{d\omega}{\omega} = -\frac{2}{3}\frac{d\tau}{\tau}$$ τ orbital period

$$\Rightarrow \frac{d\tau/dt}{\tau} = -\frac{3}{2}\frac{dE/dt}{E} = -\frac{3}{2}\frac{L}{E} = -\frac{768}{5}\frac{\omega^6 a^5}{c^5}$$

$$= -\frac{12}{5}\frac{G^3}{c^5}\frac{M^3}{a^4}$$

- Excentricity correction (Press, Thorne)

$$\frac{d\tau/dt}{\tau} = -\frac{12}{5}\frac{G^3M^2}{c^5a^4}f(e)$$

$$f(e) = \frac{1 + 73e^2/24 + 37e^4/96}{(1-e^2)^{7/2}}$$

$$\Rightarrow \frac{d\tau}{dt} = -\frac{12}{5}\frac{G^3M^2}{c^5a^4}f(e)\,\tau$$

GR: $\frac{d\tau}{dt} = -2.403(2)\times 10^{-12}$

observation: $\frac{d\tau}{dt} = -2.40(9)\times 10^{-12}$!!!

(Taylor et al) surely GWs exist

C. Quantum Gravity

I. Fundamental constants

characterize a theory

G Newtonian gravitation

c Spacetime
special Relativity

G, c General relativity

h Quantum mechanics

h, c Quantum field theory

k_B Thermodynamics, statistical Physics

G, c, h, k_B Quantum Gravity

Hawking temperature
(Black hole temperature)
$$T_H = \left(\frac{\hbar c^3}{G k_B}\right)\frac{1}{8\pi M}$$

first and up to now unique formula uniting the fundamental constants

II. Speculations

- Hawking radiation - a transplanckian phenomenum (due to redshift)
- ptle collision at LHC. If large extra dimension scenario is correct, also.
- $l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}$ cm $\qquad m_{Pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{-5}$ g $\sim 10^{16}$ TeV
- Quantum gravity no lengths, no scales
  $\Rightarrow$ topological theory